



## Alternative Axiomatic Characterizations of the Grey Shapley Value

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### Abstract

The Shapley value, one of the most common solution concepts of cooperative game theory is defined and axiomatically characterized in different game-theoretic models. Certainly, the Shapley value can be used in interesting sharing cost/reward problems in the Operations Research area such as connection, routing, scheduling, production and inventory situations. In this paper, we focus on the Shapley value for cooperative games, where the set of players is finite and the coalition values are interval grey numbers. The central question in this paper is how to characterize the grey Shapley value. In this context, we present two alternative axiomatic characterizations. First, we characterize the grey Shapley value using the properties of efficiency, symmetry and strong monotonicity. Second, we characterize the grey Shapley value by using the grey dividends.

**Keywords:** Cooperative games; uncertainty; grey numbers; the Shapley value; dividends; decision making; Operations Research.

### 1. Introduction

The grey system theory, established by Deng, (1982), is a new methodology that focuses on the study of problems involving small samples and incomplete information. In the natural world, uncertain systems with small samples and incomplete information exist commonly. That fact determines the wide range of applicability of grey system theory (Sifeng et al., 2011). In real life situations, potential rewards/costs are not known exactly and these costs often change slightly from one period to another. For instance, ordering costs for a specific product being dependent on the transportation facilities might also vary from time to time. Changes in price of oil, mailing and telephone charges may also vary the ordering costs. Therefore, grey system theory, rather than the traditional probability theory and fuzzy

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set theory, is better suited to model the sharing of reward/cost problems by using cooperative game theory (Dror and Hartman, 2011, Olgun and Alparslan Gök, 2013, Palanci et al., 2014).

The Shapley value, which is proposed by Lloyd Shapley in his 1953 Ph.D. dissertation, is one of the most common single-valued solution concepts in cooperative game theory. It is characterized for cooperative games with a finite set and where coalition values are real numbers (Shapley, 1971, Aumann and Hart, 2002, Roth, 1988). In the sequel, the Shapley value has captured much attention being extended in new game theoretic models and widely applied for solving reward/cost sharing problems the Operations Research and other related fields (see Baker, 1965, Borm et al., 2001, Curiel, 1989, Dror and Hartman, 2011, Littlechild et al. 1977, Meca, 2007, Meca et al., 2004, Mosquera et al., 2008, Alparslan Gök, 2009). Uncertainty on coalition values, which is a challenge in the real life problems, lead to new models of cooperative games and corresponding Shapley-like values (Alparslan Gök et al., 2009, 2010, 2012, Branzei et al., 2003, Timmer et al. 2003, Liao, 2012). In this study, we provide alternative axiomatic characterizations of the grey Shapley value on an additive cone of cooperative grey games inspired by the Shapley's axiomatic characterization (see Alparslan Gök et al., 2010, Derks and Peters, 1993, Shapley, 1971, Young, 1985).

This paper focuses on the properties of the grey Shapley value and axiomatically characterizes it on an additive cone of cooperative grey games<sup>1</sup>. Now, we recall the definition of a cooperative grey game (Palanci et al., 2014).

A cooperative grey game is an ordered pair  $\langle N, w' \rangle$  where  $N = \{1, \dots, n\}$  is the set of players, and  $w' = \otimes : 2^N \rightarrow \mathcal{G}(\mathbb{R})$  is the characteristic function such that  $w'(\emptyset) = \otimes_{\emptyset} \in [0, 0]$ . We denote by  $\mathcal{G}(\mathbb{R})$  the set of interval grey numbers in  $\mathbb{R}$ . The grey payoff function  $w'(S) = \otimes_S \in [\underline{A}_S, \overline{A}_S]$  refers to the value of the grey expectation benefit belonging to a coalition  $S \in 2^N$ , where  $\overline{A}_S$  and  $\underline{A}_S$  represent the maximum and minimum possible profits of the coalition  $S$ . So, a cooperative grey game can be considered as a cooperative game with grey profits  $\otimes$ . Grey solutions are useful to solve reward/cost sharing problems with grey data using cooperative grey games as a tool. Building blocks for grey solutions are grey payoff vectors, i.e. vectors whose components belong to  $\mathcal{G}(\mathbb{R})$ . We denote by  $\mathcal{G}(\mathbb{R})^N$  the set of all such grey payoff vectors, and  $\mathcal{GG}^N$  the family of all cooperative grey games.

To the best of our knowledge, no study exists on axiomatic characterization of the grey Shapley value. From this point of view, this study is a pioneering work on a promising topic. In the sequel, there are two main contributions in this paper. One of them is the characterization of the grey Shapley value with the aid of the properties of efficiency, symmetry and strong monotonicity. The other one provides a different characterization for the grey Shapley value by using the grey dividends.

The paper is organized as follows. In Section 2, we review basic notions and facts from the theory of cooperative grey games. We introduce the strong monotonicity property of the grey Shapley value on the class grey size monotonic games and give an axiomatic characterization of the grey Shapley value on a special subclass of cooperative grey games in Section 3. Section 4 provides the grey dividends and gives a characterization of the grey Shapley value by using them. The final section gives an overview, some remarks and a note on further research.

## 2. Preliminaries

This section provides some preliminaries from grey calculus, the theory of cooperative grey games and

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<sup>1</sup> We inspired from (Alparslan Gök, 2012).

the grey Shapley value (Branzei et al., 2003, Liao, 2012, Palanci et al., 2014).

A number denoted by  $\otimes \in [\underline{a}, \bar{a}]$ , where  $\underline{a}$  is called the lower limit and  $\bar{a}$  is called the upper limit for  $\otimes$  is called an interval grey number. Now, we give some operations on interval grey numbers. Let

$$\otimes_1 \in [a, b], a < b \text{ and } \otimes_2 \in [c, d], c < d.$$

The sum of  $\otimes_1$  and  $\otimes_2$ , written  $\otimes_1 + \otimes_2$ , is defined as follows:

$$\otimes_1 + \otimes_2 \in [a + c, b + d].$$

Assume that  $\otimes \in [a, b]$ ,  $a < b$ , and  $k$  is a positive real number. The scalar multiplication of  $k$  and  $\otimes$  is defined as follows:

$$k\otimes \in [ka, kb].$$

Let  $\otimes_1, \otimes_2 \in \mathcal{G}(\mathbb{R})$  with  $\otimes_1 \in [a, b]$ ,  $a < b$ ;  $\otimes_2 \in [c, d]$ ,  $c < d$ ,  $|\otimes_1| = b - a$  and  $\alpha \in \mathbb{R}_+$ . Then,

$$(i) \otimes_1 + \otimes_2 \in [a + c, b + d];$$

$$(ii) \alpha\otimes = [\alpha a, \alpha b].$$

By (i) and (ii) we see that  $\mathcal{G}(\mathbb{R})$  has a cone structure.

In this study, we examine some interval calculus from the theory of cooperative interval games. We define  $\otimes_1 - \otimes_2$ , only if  $|b - a| \geq |d - c|$ , by  $\otimes_1 - \otimes_2 \in [a - c, b - d]$ . Note that  $a - c \leq b - d$ . We recall that  $[a, b]$  is weakly better than  $[c, d]$ , which we denote by  $[a, b] \succsim [c, d]$ , if and only if  $a \geq c$  and  $b \geq d$ . We also use the reverse notation  $[a, b] \precsim [c, d]$ , if and only if  $a \leq c$  and  $b \leq d$  (for details see (Alparslan Gök et al., 2009, 2011, Branzei et al., 2003)).

Now, we recall the definition of a cooperative interval game and some basic notions from the theory of cooperative interval games. (Alparslan Gök et al., 2009, 2010). A cooperative interval game is an ordered pair  $\langle N, w \rangle$  where  $N = \{1, \dots, n\}$  is the set of players, and  $w: 2^N \rightarrow I(\mathbb{R})$  is the characteristic function such that  $w(\emptyset) = [0, 0]$ . Here,  $I(\mathbb{R})$  is the set of all nonempty, compact intervals in  $\mathbb{R}$ . For each  $S \in 2^N$ , the worth set (or worth interval)  $w(S)$  of the coalition  $S$  in the interval game  $\langle N, w \rangle$  is of the form  $[\underline{w}(S), \bar{w}(S)]$ , where  $\underline{w}(S)$  is the lower bound and  $\bar{w}(S)$  is the upper bound of  $w(S)$ . The family of all interval games with player set  $N$  is denoted by  $IG^N$ . Note that, if all the worth intervals are degenerate intervals, i.e.  $\underline{w}(S) = \bar{w}(S)$  for each  $S \in 2^N$ , then the interval game  $\langle N, w \rangle$  corresponds in a natural way to the classical cooperative game  $\langle N, v \rangle$  where  $v(S) = \underline{w}(S)$  for all  $S \in 2^N$ . Some classical cooperative games associated with an interval game  $w \in IG^N$  have played a key role, namely the border games  $\langle N, \underline{w} \rangle$ ,  $\langle N, \bar{w} \rangle$  and the length

game  $\langle N, |w| \rangle$ , where  $|w|(S) = \overline{w}(S) - \underline{w}(S)$  for each  $S \in 2^N$ . Note that  $\overline{w} = \underline{w} + |w|$ .

Next, we recall that the definition of the grey Shapley value and the properties of the grey Shapley value (Palanci et al., 2014).

For  $w, w_1, w_2 \in IG^N$  and  $w'_1, w'_2 \in \mathcal{GG}^N$  we say that  $w'_1 \in w_1 \leq w'_2 \in w_2$  if  $w'_1(S) \leq w'_2(S)$  where  $w'_1(S) \in w_1(S)$  and  $w'_2(S) \in w_2(S)$  and, for each  $S \in 2^N$ .

For  $w'_1, w'_2 \in \mathcal{GG}^N$  and  $\lambda \in \mathbb{R}_+$  we define  $\langle N, w'_1 + w'_2 \rangle$  and  $\langle N, \lambda w' \rangle$  by  $(w'_1 + w'_2)(S) = w'_1(S) + w'_2(S)$  and  $(\lambda w')(S) = \lambda w'(S)$  for each  $S \in 2^N$ . So, we conclude that  $\mathcal{GG}^N$  endowed with  $\leq$  is a partially ordered set and has a cone structure with respect to addition and multiplication with non-negative scalars described above. For  $w'_1, w'_2 \in \mathcal{GG}^N$  where  $w'_1 \in w_1, w'_2 \in w_2$  with  $|w_1(S)| \geq |w_2(S)|$  for each  $S \in 2^N$ ,  $\langle N, w'_1 - w'_2 \rangle$  is defined by  $(w'_1 - w'_2)(S) = w'_1(S) - w'_2(S) \in w_1(S) - w_2(S)$ .

We call a game  $\langle N, w' \rangle$  *grey size monotonic* if  $\langle N, |w| \rangle$  is monotonic, i.e.,  $|w|(S) \leq |w|(T)$  for all  $S, T \in 2^N$  with  $S \subset T$ . For further use we denote by  $SM\mathcal{GG}^N$  the class of all grey size monotonic games with player set  $N$ .

The grey marginal operators and the grey Shapley value are defined on  $SM\mathcal{GG}^N$ . Denote by  $\Pi(N)$  the set of permutations  $\sigma: N \rightarrow N$  of  $N$ . The *grey marginal operator*  $m^\sigma: SM\mathcal{GG}^N \rightarrow \mathcal{G}(\mathbb{R})^N$  corresponding to  $\sigma$ , associates with each  $w' \in SM\mathcal{GG}^N$  the *grey marginal vector*  $m^\sigma(w')$  of  $w'$  with respect to  $\sigma$  defined by

$m_i^\sigma(w') = w'(P^\sigma(i) \cup \{i\}) - w'(P^\sigma(i)) \in [\underline{A}_{P^\sigma(i) \cup \{i\}} - \underline{A}_{P^\sigma(i)}, \overline{A}_{P^\sigma(i) \cup \{i\}} - \overline{A}_{P^\sigma(i)}]$  for each  $i \in N$ , where  $P^\sigma(i) = \{r \in N \mid \sigma^{-1}(r) < \sigma^{-1}(i)\}$ , and  $\sigma^{-1}(i)$  denotes the entrance number of player  $i$ . For grey size monotonic games  $\langle N, w' \rangle$ ,  $w'(T) - w'(S) \in w(T) - w(S)$  is defined for all  $S, T \in 2^N$  with  $S \subset T$  since  $|w(T)| = |w|(T) \geq |w|(S) = |w|(S)|$ . Now, we notice that for each  $w' \in SM\mathcal{GG}^N$  the grey marginal vectors  $m^\sigma(w')$  are defined for each  $\sigma \in \Pi(N)$ , because the monotonicity of  $|w|$  implies  $\overline{A}_{S \cup \{i\}} - \underline{A}_{S \cup \{i\}} \geq \overline{A}_S - \underline{A}_S$ , which can be rewritten as  $\overline{A}_{S \cup \{i\}} - \overline{A}_S \geq \underline{A}_{S \cup \{i\}} - \underline{A}_S$ . So,  $w'(S \cup \{i\}) - w'(S) \in w(S \cup \{i\}) - w(S)$  is defined for each  $S \subset N$  and  $i \notin S$ . We notice that all the grey marginal vectors of a grey size monotonic game are efficient grey payoff vectors. for each  $w' \in SM\mathcal{GG}^N$ .

We call a map  $\Psi': SM\mathcal{GG}^N \rightarrow \mathcal{G}(\mathbb{R})^N$  a solution on  $SM\mathcal{GG}^N$ .

A solution  $\Psi'$  satisfies additivity if  $\Psi'(w'_1 + w'_2) = \Psi'(w'_1) + \Psi'(w'_2)$  for all  $w'_1, w'_2 \in SM\mathcal{GG}^N$ .

**Proposition 2.1** The grey Shapley value  $\Phi': SM\mathcal{GG}^N \rightarrow \mathcal{G}(\mathbb{R})^N$  is additive.

Let  $w' \in SM\mathcal{GG}^N$  and  $i, j \in N$ . Then,  $i$  and  $j$  are called symmetric players, if  $w'(S \cup \{j\}) - w'(S) = w'(S \cup \{i\}) - w'(S)$ , for each  $S$  with  $i, j \notin S$ .

A solution  $\Psi'$  satisfies symmetry property if  $\Psi'_i(w) = \Psi'_j(w)$  for all  $w' \in SMGG^N$  and all symmetric players  $i$  and  $j$  in  $w$ .

**Proposition 2.2** Let  $i, j \in N$  be symmetric players in  $w' \in SMGG^N$ . Then,  $\Phi'_i(w') = \Phi'_j(w')$ .

Let  $w' \in SMGG^N$  and  $i \in N$ . Then,  $i$  is called a *dummy player* if  $w'(S \cup \{i\}) = w'(S) + w'(\{i\})$ , for each  $S \in 2^{N \setminus \{i\}}$ .

A solution  $\Psi'$  satisfies dummy player property if  $\Psi'_i(w') = w'(\{i\})$  for all  $w' \in SMGG^N$  and all dummy players  $i$  in  $w'$ .

**Proposition 2.3** The grey Shapley value  $\Phi' : SMGG^N \rightarrow \mathcal{G}(\mathbb{R})^N$  has the dummy player property, i.e.  $\Phi'_i(w') = w'(\{i\})$  for all  $w' \in SMGG^N$  and for all dummy players  $i$  in  $w'$ .

A solution  $\Psi'$  satisfies efficiency if  $\sum_{i=1}^n \Psi'_i(w) = w'(N)$  for all  $w' \in SMGG^N$ .

**Proposition 2.4** The grey Shapley value  $\Phi' : SMGG^N \rightarrow \mathcal{G}(\mathbb{R})^N$  is efficient, i.e.,  $\sum_{i \in N} \Phi'_i(w') = w'(N)$ .

### 3. Research Literature

In this section, some literature leading to grey system theory and the theory of cooperative games is provided in historical order. Shapley (1953) introduced the Shapley value which is one of the common solution concepts in cooperative game theory. Deng (1982) studied the stability and stabilization of a grey system whose state matrix is triangular. Furthermore, the displacement operator and established transfer developed by the author are the indispensable tool for the grey system theory. Borm et al. (2001) made a survey on the research area of cooperative games associated with several types of Operations Research problems in which various decision makers (players) are involved. Liu and Lin (2006) published a book in which they extensively described the grey system theory and its applications. Jahanshahloo et al. (2006) investigated the problems of consensus-making among individuals or organizations with multiple criteria for evaluating performance when the players are supposed to be egoistic and the score for each criterion for a player is supposed to be an interval. Fang et al. (2006) studied the pure strategy of the grey game and its solution concept called the grey pure strategy solution. Liu and Kao (2009) developed a solution method for the two-person zero-sum game, where the payoffs are imprecise and represented by interval data. Alparslan Gök et al. (2009) introduced the class of convex interval games and extended classical results regarding the characterizations of convex games and the properties of solution concepts to the interval setting. Branzei et al. (2010) briefly presented the state-of-the-art of the cooperative interval games. They discussed how the model of cooperative interval games extends the cooperative game theory literature, and reviewed its existing and potential applications in economic and Operations Research situations with interval data. Sifeng et al. (2011) introduced the elementary concepts and the fundamental principles of grey systems, and the main components of grey system theory. Alparslan Gök (2012)

characterized the Shapley value by using interval calculus. Olgun and Alparslan Gök (2013) introduced the cooperative grey games and focused on Sharing Ordering Cost rule (SOC-rule) to distribute the joint cost. Palanci et al. (2014) characterized the Shapley value by using the grey numbers.

#### 4. Strong monotonicity and a characterization

In Palanci et al., 2014, it is shown that the grey Shapley value satisfies additivity, efficiency, symmetry and dummy player properties on the class of grey size monotonic games. Now, we give another property of the grey Shapley value on the same class and show that this property together with efficiency and symmetry characterize the grey Shapley value on a cone of grey games which is the main novelty of this study.

A solution  $\Psi'$  satisfies strong monotonicity if  $\Psi'_i(w_1) \geq \Psi'_i(w_2)$  for all  $w'_1, w'_2 \in SMGG^N$  that satisfy  $w'_1(S \cup \{i\}) - w'_1(S) \geq w'_2(S \cup \{i\}) - w'_2(S)$  for all  $S \in 2^N$ , where  $w'_1(S \cup \{i\}) - w'_1(S) \succ w'_2(S \cup \{i\}) - w'_2(S)$  and  $w'_1 \in w_1, w'_2 \in w_2$ .

**Proposition 4.1** *The grey Shapley value  $\Phi' : SMGG^N \rightarrow \mathcal{G}(\mathbb{R})^N$  is strong monotonic<sup>1</sup>, i.e. if  $w'_1(S \cup \{i\}) - w'_1(S) \geq w'_2(S \cup \{i\}) - w'_2(S)$  for all  $S \in 2^N$ , where  $w'_1(S \cup \{i\}) - w'_1(S) \succ w'_2(S \cup \{i\}) - w'_2(S)$ , and  $w'_1 \in w_1, w'_2 \in w_2$ , then  $\Phi'_i(w'_1) \geq \Phi'_i(w'_2)$  for all  $w'_1, w'_2 \in SMGG^N$ .*

**Proof.** The proof is immediate from (3).

We can say that the grey Shapley value has the property that if a player contributes at least as much to any coalition in a game  $w'_1 \in SMGG^N$  than in a game  $w'_2 \in SMGG^N$  then his/her payoff from the grey Shapley value in  $w'_1$  is at least as large as that in  $w'_2$ .

Let  $S \in 2^N \setminus \{\emptyset\}$ ,  $\otimes \in \mathcal{G}(\mathbb{R})$  and let  $u_S$  be the classical unanimity game based on  $S$  (see (Moore, 1979)). The cooperative interval game  $\langle N, \otimes u_S \rangle$  is defined by  $(\otimes u_S)(T) = \otimes u_S(T)$  for each  $T \in 2^N \setminus \{\emptyset\}$ , and its Shapley value is given by

$$\Phi'_i(\otimes u_S) = \begin{cases} \otimes / |S|, & i \in S \\ [0, 0], & i \notin S. \end{cases}$$

We denote by  $KGG^N$  the additive cone generated by the set

$$K = \{\otimes_S u_S \mid S \in 2^N \setminus \{\emptyset\}, \otimes_S \in \mathcal{G}(\mathbb{R})\}.$$

So, each element of the cone is a finite sum of elements of  $K$ .

**Remark 4.1** We notice that  $KGG^N \subset SMGG^N$ , and axiomatically characterize the restriction of the grey Shapley value to the cone  $KGG^N$ . That is, all the properties of the grey Shapley value are inherited on  $KGG^N$ .

<sup>1</sup> Strong monotonicity property is introduced by (Young, 1985) for the crisp case.

**Theorem 4.1**<sup>1</sup> There is a unique solution  $\Psi' : KGG^N \rightarrow \mathcal{G}(\mathbb{R})^N$  satisfying the properties of efficiency, symmetry and strong monotonicity. This solution is the grey Shapley value.

**Proof.**<sup>2</sup> From Propositions 2.2, 2.4, 3.1 and  $KGG^N \subset SMGG^N$  we obtain that  $\Phi'$  satisfies the three properties on  $KGG^N$ .

Conversely, let  $\Psi'$  be an interval value satisfying the three properties on  $KGG^N$ . We have to show that  $\Psi' = \Phi'$ .

(i) Let  $\tilde{w}'$  be the game which is identically an element of a zero interval. In this game, all players are symmetric, so symmetry and efficiency together imply  $\Psi'(\tilde{w}') \in ([0,0], \dots, [0,0])$ .

(ii) Let  $i$  be a dummy player in the game  $\tilde{w}'$ . If the condition in strong monotonicity applied to  $\tilde{w}'$  and  $w'$  with all inequalities being equalities then by strong monotonicity we have  $\Psi'_i(w') \geq \Psi'_i(\tilde{w}')$  and  $\Psi'_i(\tilde{w}') \geq \Psi'_i(w')$ . Hence by (i),  $\Psi'_i(w') \in [0,0]$ .

(iii) Let  $\otimes \in \mathcal{G}(\mathbb{R})$  and  $T \in 2^N \setminus \{\emptyset\}$ . Then (ii) implies  $\Psi'_i(\otimes u_T) \in [0,0]$  for all  $i \in N \setminus T$ , and symmetry and efficiency imply  $\Psi'_i(\otimes u_T) = \frac{\otimes}{|T|}$  for every  $i \in T$ . Hence,  $\Psi'(\otimes u_T) = \frac{\otimes}{|T|} e^T$ .

(iv) Each  $w' \in KGG^N$  can be written as  $w' = \sum_{T \in 2^N \setminus \{\emptyset\}} \otimes_T u_T$  by definition.

To prove  $\Psi'(w') = \Phi'(w')$  we use induction on the number  $\alpha(w')$  of terms in  $\sum_{T \in 2^N \setminus \{\emptyset\}} \otimes_T u_T$  with  $\otimes_T \notin [0,0]$ .

If  $\alpha(w') = 0$  then by (i),  $\Psi'(w') = \Phi'(w') \in ([0,0], \dots, [0,0])$  and if  $\alpha(w') = 1$  then by (iii),  $\Psi'(w') = \Phi'(w')$  because  $\Phi'(\otimes u_T) = \frac{\otimes}{|T|} e^T$ .

Assume that  $\Psi'(\tilde{w}') = \Phi'(\tilde{w}')$  for all  $\tilde{w}' \in KGG^N$  with  $\alpha(\tilde{w}') < k$ , where  $k \geq 2$ . Let  $w'$  be a game with  $\alpha(w') = k$ . Then there are coalitions  $T_1, T_2, \dots, T_k$  and interval grey numbers  $\otimes_1, \otimes_2, \dots, \otimes_k \notin [0,0]$ , such that  $w' = \sum_{r=1}^k \otimes_r u_{T_r}$ . Let  $D := \cap_{r=1}^k T_r$ .

For  $i \in N \setminus D$ , define  $\tilde{w}^i := \sum_{r: i \in T_r} \otimes_r u_{T_r}$ . The induction hypothesis implies that  $\Psi'_i(\tilde{w}^i) = \Phi'_i(\tilde{w}^i)$  because  $\alpha(\tilde{w}^i) < k$ . Further, for every  $S \in 2^N$ ,

$$w'(S \cup \{i\}) - w'(S) = \sum_{r=1}^k \otimes_r u_{T_r}(S \cup \{i\}) - \sum_{r=1}^k \otimes_r u_{T_r}(S)$$

<sup>1</sup> This characterization is a straightforward generalization of Young, 1985.

<sup>2</sup> We follow the steps of Peters, 2008.

$$\begin{aligned}
 &= \sum_{r:i \in T_r} \otimes_r u_{T_r}(S \cup \{i\}) - \sum_{r:i \in T_r} \otimes_r u_{T_r}(S) \\
 &= \tilde{w}^i(S \cup \{i\}) - \tilde{w}^i(S).
 \end{aligned}$$

Therefore by strong monotonicity of  $\Psi'$  and  $\Phi'$  we have

$$\Psi'_i(w') = \Psi'_i(\tilde{w}^i) = \Phi'_i(\tilde{w}^i) = \Psi'_i(w') \text{ implying that}$$

$$\Psi'_i(w') = \Phi'_i(w') \text{ for all } i \in N \setminus D. \quad (1)$$

Equation (1) and efficiency for  $\Psi'$  and  $\Phi'$  implies

$$\sum_{i \in D} \Psi'_i(w') = \sum_{i \in D} \Phi'_i(w'). \quad (2)$$

Let  $i, j \in D$ , then for every  $S \subset N \setminus \{i, j\}$  we have

$$w'(S \cup \{i\}) = \sum_{r=1}^k \otimes_r u_{T_r}(S \cup \{i\}) = \sum_{r=1}^k \otimes_r u_{T_r}(S \cup \{j\}) = w'(S \cup \{j\}) \in [0, 0],$$

so  $i$  and  $j$  are symmetric.

Hence, by symmetry of  $\Phi'$  and  $\Psi'$  we have

$$\Psi'_i(w') = \Psi'_j(w'), \Phi'_i(w') = \Phi'_j(w'). \quad (3)$$

Finally  $\Phi'(w') = \Psi'(w')$  follows from (1), (2) and (3) which completes the proof.

By the above theorem we see that the strong monotonicity provides a simple characterization of the grey Shapley value without resorting to the usual additivity and dummy player properties.

## 5. Grey dividends and a characterization

In this section we give a different characterization for the grey Shapley value in terms of grey dividends. The grey dividends<sup>1</sup> for each nonempty coalition  $T$  in a game  $w' \in SMGG^N$  are defined recursively as follows:

$$\begin{aligned}
 d'(w') &:= w'(T) = \otimes_T \in [\underline{A}_T, \overline{A}_T] \text{ for all } T \text{ with } |T| = 1, \\
 d'(w') &:= \frac{w'(T) - \sum_{S \subset T, S \neq T} |S| d'_S(w')}{|T|} \in \left[ \frac{w'(T) - \sum_{S \subset T, S \neq T} |S| \underline{A}_S(w')}{|T|}, \frac{w'(T) - \sum_{S \subset T, S \neq T} |S| \overline{A}_S(w')}{|T|} \right] \text{ if } |T| > 1.
 \end{aligned}$$

We notice that for each  $w' \in SMGG^N$  the grey dividends are defined with a similar manner as the grey marginal vectors.

The relation between the grey dividends and the grey Shapley value is described in the next theorem. The grey Shapley value of a player in a grey game turns out to be the sum of all equally distributed grey dividends of coalitions to which the player belongs.

**Proposition 5.1** *Let  $w' \in KGG^N$  and*

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<sup>1</sup> The definition of interval dividends is a generalization of Peters, 2008.



$$w' = \sum_{T \in 2^N \setminus \{\emptyset\}} d'_T u_T \in \sum_{T \in 2^N \setminus \{\emptyset\}} [A_T u_T, \sum_{T \in 2^N \setminus \{\emptyset\}} \overline{A}_T u_T].$$

Then

- (i)  $|T| d'_T(w') = \otimes_T$  for all  $T \in 2^N \setminus \{\emptyset\}$ ,
- (ii)  $\Phi'_i(w') = \sum_{T:i \in T} d'_T(w')$  for all  $i \in N$ .

**Proof.** The proof is a straightforward generalization from the classical case and can be obtained by following the steps of Theorem 17.7 in (Peters, 2008).

The following examples illustrate constructing the model and calculating the grey Shapley value by using Proposition 4.1. Example 5.1 from (Branzei et al., 2010) illustrates a real life application from an OR situation.

**Example 5.1**<sup>1</sup> Let  $\langle N, w \rangle$  be a cooperative grey game with  $N = \{1, 2, 3\}$  and  $w'(\{1\}) = w'(\{1, 3\}) \in [7, 7]$ ,  $w'(\{1, 2\}) \in [7 + 5, 7 + 10]$ ,  $w'(\{1, 2, 3\}) \in [7 + 5 + 12, 7 + 10 + 12]$ , and  $w'(S) \in [0, 0]$  in any other case. The grey Shapley value of this game can be calculated as follows:

$$d'_{\{1\}}(w') \in [7, 7], d'_{\{2\}}(w') = d'_{\{3\}}(w') \in [0, 0], d'_{\{1,2\}}(w') \in [2\frac{1}{2}, 5], d'_{\{1,3\}} = d'_{\{2,3\}} \in [0, 0], d'_N(w') \in [4, 4].$$

$$\Phi'_1(w') = \sum_{T:1 \in T} d'_T(w') \in [\frac{27}{2}, 16], \quad \Phi'_2(w') = \sum_{T:2 \in T} d'_T(w') \in [\frac{13}{2}, 9],$$

$$\Phi'_3(w') = \sum_{T:3 \in T} d'_T(w') \in [4, 4]. \text{ So, } \Phi'(w') \in ([13\frac{1}{2}, 16], [6\frac{1}{2}, 9], [4, 4]).$$

**Example 5.2**<sup>2</sup> Let  $\langle N, w' \rangle$  be a cooperative grey game with  $N = \{1, 2, 3\}$  and  $w'(\emptyset) \in [0, 0]$ ,  $w'(1) = w'(2) = w'(3) \in [0, 0]$ ,  $w'(1, 2) = w'(1, 3) = w'(2, 3) \in [2, 4]$  and  $w'(N) \in [9, 15]$ . The grey Shapley value of this game can be calculated as follows:

$$d'_{\emptyset}(w') = d'_1(w') = d'_2(w') = d'_3(w') \in [0, 0]$$

$$d'_{1,2}(w') = d'_{1,3}(w') = d'_{2,3}(w') \in [1, 2], d'_N(w') \in [1, 1].$$

$$\Phi'_1(w') = \sum_{T:1 \in T} d'_T(w') = d'_1(w') + d'_{1,2}(w') + d'_{1,3}(w') + d'_N(w') \in [1, 2] + [1, 2] + [1, 1] = [3, 5],$$

$$\Phi'_2(w') = \sum_{T:2 \in T} d'_T(w') = d'_2(w') + d'_{1,2}(w') + d'_{2,3}(w') + d'_N(w') \in [1, 2] + [1, 2] + [1, 1] = [3, 5],$$

$$\Phi'_3(w') = \sum_{T:3 \in T} d'_T(w') = d'_3(w') + d'_{1,3}(w') + d'_{2,3}(w') + d'_N(w') \in [1, 2] + [1, 2] + [1, 1] = [3, 5].$$

Hence,  $\Phi'(w') \in ([3, 5], [3, 5], [3, 5])$ .

<sup>1</sup> see (Branzei et al., 2010) for a motivation.

<sup>2</sup> see also (Alparslan Gök, 2009).

## 6. Conclusion

In this study, we present two alternative axiomatic characterizations of the grey Shapley value. One of the characterizations is provided by using the properties of efficiency, symmetry and strong monotonicity, and the other one is done by using the grey dividends.

We notice that whereas the Shapley value is defined and axiomatically characterized for arbitrary cooperative games, the grey Shapley value is defined only for a subclass of cooperative grey games, called grey size monotonic games. Moreover, the grey Shapley value is axiomatically characterized only on the strict subset of grey size monotonic games. The restriction to the class of size monotonic games is imposed by the need to establish efficiency of interval marginal vectors, and consequently of the grey Shapley value. For further research, the grey Shapley value can be used in interesting sharing cost/reward problems the Operations Research area.

## References

- Alparslan Gök S.Z., (2012). On the interval shapley value, *Optimization*, pp. 1-9.
- Alparslan Gök S.Z., Branzei O., Branzei R. and Tijs S., (2011). Set-valued solution concepts using interval-type payoffs for interval games, *Journal of Mathematical Economics*, Vol. 47, pp. 621-626.
- Alparslan Gök S.Z., Branzei R. and Tijs S., (2009). Airport Interval Games and Their Shapley Value, *Operations Research and Decisions*, No 2, (ISSN 1230-1868)).
- Alparslan Gök S.Z., Branzei R. and Tijs S., (2009). Convex interval games, *Journal of Applied Mathematics and Decision Sciences*, Vol. 2009, Article ID 342089, 14 pages, DOI: 10.1115/2009/342089.
- Alparslan Gök S.Z., Branzei R. and Tijs S., (2010). The interval Shapley value: an axiomatization, *Central European Journal of Operations Research (CEJOR)*, Vol. 18(2), pp. 131-140.
- Aumann R.J. and Hart S., (2002). Handbook of game theory with economic applications, 3rd edn., *Elsevier*, Amsterdam.
- Baker J.Jr., (1965). Airport runway cost impact study, *Report submitted to the Association of Local Transport Airlines*, Jackson, Mississippi.
- Borm P., Hamers H. and Hendrickx R., (2001). Operations research games: a survey. *TOP*, Vol. 9, pp. 139-216.
- Branzei R., Branzei O., Alparslan Gök S.Z. and Tijs S., (2010). Cooperative Interval Games: A Survey, *Central European Journal of Operations Research (CEJOR)*, Vol. 18(3), pp. 397-411.
- Branzei R., Dimitrov D. and Tijs S., (2003). Shapley-like values for interval bankruptcy games, *Economics Bulletin*, Vol. 3, pp. 1-8.

- Branzei R., Dimitrov D. and Tijs S., (2008). Models in Cooperative Game Theory, *Game Theory and Mathematical Methods*, Springer, Berlin, Germany.
- Branzei R., Mallozzi L. and Tijs S., (2010). Peer group situations and games with interval uncertainty, *International Journal of Mathematics, Game Theory, and Algebra*, Vol. 19(5-6), pp. 381-388.
- Curiel I., Pederzoli G. and Tijs S., (1989). Sequencing games. *European Journal of Operational Research*, Vol. 40, pp. 344-351.
- Deng J., (1982). Control problems of Grey Systems, *Systems and Control Letters*, Vol. 5, pp. 288-294.
- Derks J. and Peters H., (1993). A Shapley Value for Games with Restricted Coalitions, *International Journal of Game Theory*, Vol. 21, pp. 351-360.
- Dror M. and Hartman B.C., (2011). Survey of cooperative inventory games and extensions, *Journal of the Operational Research Society*, Vol. 62, pp. 565-580.
- Fang Z., Sifeng L., Chuanmin M., (2006), Study on optimum value of Grey Matrix Game Based on Token of Grey Interval Number, *Proceedings of the 6th world congress on Intelligent Control and Automation*, June 21-23, DLLn, China.
- Jahanshahloo G.R, Hosseinzadeh Lotfi F., Sohraiee S., (2006), Egoist's Dilemma with interval data, *Applied Mathematics and Computation*, Vol. 183, pp. 94-105.
- Liao Y.H., (2012). Alternative axiomatizations of the Shapley value under interval uncertainty, *International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems*, Vol. 20, pp. 619-628.
- Littlechild S.C. and Thompson G.F., (1977). Aircraft landing fees: A game theory approach, *The Bell Journal of Economics*, Vol. 8, No.1 , 186-204.
- Liu S. and Lin Y., (2006). Grey Information: Theory and Practical Applications, *Springer*, Germany.
- Liu S.T., Kao C., (2009), Matrix games with interval data, *Computers&Industrial Engineering*, Vol. 56, pp. 1697-1700.
- Meca A., (2007). A core-allocation family for generalized holding cost games, *Mathematical Methods of Operations Research*, Vol. 65, pp. 499-517.
- Meca A., Timmer J., Garcia-Jurado I. and Borm P., (2004). Inventory games, *European Journal of Operational Research*, Vol. 156, pp. 127-139.
- Moore R., (1979). Methods and applications of interval analysis, *SIAM*, Philadelphia.
- Mosquera M.A., Garcia-Jurado I. and Borm P., (2008). A note on coalitional manipulation and centralized inventory management, *Annals of Operations Research*, Vol. 158, pp. 183-188.

Olgun M.O. and Alparslan Gök S.Z., (2013). Cooperative Grey Games and An Application on Economic Order Quantity Model, *Institute of Applied Mathematics*, number 2013-27, Preprint,

Palanci O., Alparslan Gök S.Z., Ergun S. and Weber G.-W., (2014). Cooperative grey games and the grey Shapley value, submitted 2014.

Peters H., (2008). Game Theory: A Multi-Leveled Approach , *Springer*.

Roth A., (1988). The Shapley value: essays in honor of Lloyd Shapley, *Cambridge University Press*, Cambridge.

Shapley L.S., (1953). A value for  $n$ -person games, *Annals of Mathematics Studies*, Vol. 28, pp. 307-317.

Shapley L.S., (1971). Cores of convex games, *International Journal of Game Theory*, vol. 1, no. 1, pp. 11-26.

Sifeng L., Forrest J. and Yang Y., (2011). Brief Introduction to Grey Systems Theory, *Grey Systems and Intelligent Services (GSIS)*, 2011 IEEE International Conference, pp. 1-9.

Thompson G.F.,(1971). Airport costs and pricing, Unpublished PhD. Dis-sertation, University of Birmingham.

Timmer J., Borm P. and Tijs S., (2003). On three Shapley-like solutions for cooperative games with random payoffs, *International Journal of Game Theory*, Vol. 32, pp. 595-613.

Young H.P., (1985). Monotonic solutions of cooperative games, *International Journal of Game Theory*, Vol. 14, pp. 65-72.